On the moduli space of hypersurfaces singular along a subscheme of large dimension but small degree

Kaloyan Slavov

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Abstract

Let k be an algebraically closed field. Fix integers n and b with $n \geq 3$ and $1 \leq b \leq n-1$. Let T_k^d be the moduli space of hypersurfaces $[F]$ in $\mathbb{P}_k^n$ of degree l whose singular locus contains a subscheme of dimension b with Hilbert polynomial among the Hilbert polynomials of b-dimensional integral closed subschemes of $\mathbb{P}^n$ of degree d. We prove that when l is sufficiently large and $2 \leq d \leq \frac{l+1}{2}$, any irreducible component Z of T_k^d satisfies $Z = T_k^1$ or $\dim Z < \dim T_k^1$.

Contents

1 Introduction

Let n and b be fixed integers with $n \geq 3$ and $1 \leq b \leq n-1$, and let k be an algebraically closed field. Fix a positive integer l. Inside the projective space of all hypersurfaces

in $\mathbb{P}^n$ of degree l, consider the ones which are singular along some b-dimensional closed subscheme,

$$X = \{[F] \in \mathbb{P}(k[x_0, ..., x_n]_l) \mid \dim V(F)_{\text{sing}} \geq b\}$$

(this is a closed subset).

A simple argument (Lemma 5.1) will show that

$$X^1 := \{[F] \in X \mid L \subset V(F)_{\text{sing}} \text{ for some linear } b\text{-dimensional } L \subset \mathbb{P}^n\}$$

is an irreducible closed subset of X of dimension $\binom{l+n}{n} - a_{n,b}(l)$, where

$$\begin{aligned} a_{n,b}(l) &:= \binom{l+b}{b} + (n-b)\binom{l-1+b}{b} + 1 - (b+1)(n-b) \\ &= \frac{n-b+1}{b!} l^b + \dots . \end{aligned}$$

Define T_k^d as the closed subset of $\mathbb{P}(k[x_0, ..., x_n]_l)$ consisting of all hypersurfaces $[F]$ such that $V(F)_{\text{sing}}$ contains a b-dimensional closed subscheme whose Hilbert polynomial is among the Hilbert polynomials of integral b-dimensional closed subschemes of degree d. Note that $T_k^1 = X^1$. The goal of this paper is to prove the following

Theorem 1.1. *There exists $l_0 = l_0(n, b)$ (easily computable) such that for all pairs (d, l) with $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$, the following holds: if $Z \subset T_k^d$ is an irreducible component, then either $Z = X^1$, or $\dim Z < \dim X^1$.*

This is the first step ("case of small degree d") towards the theorem below, which will be proved in a subsequent paper ([6]):

Theorem 1.2. *There exists an integer $l_0 = l_0(n, b, char\ k)$, such that for all $l \geq l_0$, X^1 is the unique irreducible component of X of maximal dimension.*

In the proof of Theorem 1.1, we assume a conjecture by Eisenbud and Harris in the case $b \geq 2$. The proof of Theorem 1.1 will give a simple procedure to compute a possible value of l_0, given n and b. In addition, in this paper, we prove a result analogous to Theorem 1.1 but regarding the second largest component of X. Again in [6], we will use this result to show that for large l, the second largest component of X comes from the hypersurfaces singular along an integral closed subscheme of degree 2, at least when $\text{char} k > 0$.

We now sketch the main idea of the proof. Let Hilb^d denote the disjoint union of the finitely many Hilbert schemes $\text{Hilb}_{\mathbb{P}^n}^{P_\alpha}$, where P_α ranges over the Hilbert polynomials of integral b-dimensional closed subschemes $C \subset \mathbb{P}^n$ of degree d, and define the restricted Hilbert scheme $\widetilde{\text{Hilb}}^d$ as the closure in Hilb^d of the set of points corresponding to integral subschemes. Let $V = k[x_0, ..., x_n]_l$. Consider the incidence correspondence

$$\widetilde{\Omega}^d = \{(C, [F]) \in \widetilde{\text{Hilb}}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}}\}.$$

We will show[1] that for $2 \leq d \leq \frac{l+1}{2}$ ("small" degree), any irreducible component of $\widetilde{\Omega}^d$ has dimension less than $\dim X^1$. For this, we apply the theorem on dimension of fibers to

[1]We are going to be slightly imprecise here; see Section 5.3 for the exact statement.

the map $\pi\colon \widetilde{\Omega}^d \to \widetilde{\mathrm{Hilb}}^d$. A result of Eisenbud and Harris gives $\dim \widetilde{\mathrm{Hilb}}^d$ when $b = 1$; for $b > 1$, they state a conjecture for the corresponding result. (We assume this conjecture but also note that our proof can be modified to give an alternative unconditional — but ineffective — proof of a weaker version of Theorem 1.1 that will still suffice for Theorem 1.2.) So it remains to give an upper bound for the dimension of the fiber of π over an integral C of degree d. For this, we specialize C to a union of d b-dimensional linear subspaces that contain a common $(b-1)$-dimensional linear subspace.

2 Notation

For a field k, the graded ring $k[x_0, ..., x_n]$ will be denoted by S. For a graded S-module M (in particular, for a homogeneous ideal), M_l will denote the l-th graded piece of M. When $I \subset S$ is a homogeneous ideal, $(I^2)_l$ is denoted simply by I_l^2. When the field k and the integer l are fixed, V will denote the vector space $V = k[x_0, ..., x_n]_l$.

For a finite-dimensional k-vector space V, $\mathbb{P}(V)$ denotes the projective space parametrizing lines in V, so for a k-scheme S, $\mathrm{Hom}_{\mathrm{Sch}\,/k}(S, \mathbb{P}(V))$ consists of a line bundle Ł on S, together with an injective bundle map (i.e., with locally free cokernel) $Ł \hookrightarrow V \otimes_k Ø_S$. Given a homogeneous ideal $I \subset k[x_0, ..., x_n]$, $V(I)$ denotes the closed subscheme $\mathrm{Proj}(k[x_0, ..., x_n]/I) \hookrightarrow \mathbb{P}_k^n$, and for $i = 0, ..., n$, $D_+(x_i)$ is the complement of $V(x_i)$. We often abbreviate $V(\{G_i\}_{i \in I}) \subset \mathbb{P}^n$ as $V(G_i)$, when the index set I is irrelevant or understood.

For $F \in S_l$, $V(F)_{\mathrm{sing}} \subset \mathbb{P}^n$ is the closed subscheme $V(F, \frac{\dot{F}}{\dot{X}_i}) = V(F, \frac{\dot{F}}{\dot{X}_0}, ..., \frac{\dot{F}}{\dot{X}_n})$ of $\mathbb{P}^n$, so when $F \neq 0$, the underlying topological space of $V(F)_{\mathrm{sing}}$ is the singular locus of $V(F)$.

If $C \hookrightarrow \mathbb{P}^n$ is a closed subscheme of dimension b and Hilbert polynomial $P_C(z) = \frac{d}{b!} z^b + \ldots$, we say that C has degree d.

We will reuse l_0 for different bounds as we go along, in order to avoid unnecessary notation; however, it will be clear that we are actually referring to different values of l_0 even though we use the same symbol. Also, it will be understood that sometimes the value of l_0 is the maximum of a finite set of previously defined bounds, each of them still denoted by l_0.

When X is a scheme of finite type over an algebraically closed field, we often identify X with its set of closed points, since most of our arguments will be just on the level of closed points. So when we say "$x \in X$," we usually refer to a closed point $x \in X$ (this will be clear from the context).

3 The incidence correspondence

The goal of this section is to prove that the incidence correspondence is a closed subset of the product $\mathrm{Hilb}^P \times \mathbb{P}(k[x_0, ..., x_n]_l)$ (Corollary 3.2) and to define the moduli spaces $T^P \to \mathrm{Spec}\,\mathbb{Z}$ (defined at the end of the section). For the sake of the proof of just Theorem 1.1, it would suffice to carry the discussion of this section over $\mathrm{Spec}\,k$. However, the reason we want to work in the universal setting over $\mathrm{Spec}\,\mathbb{Z}$ is that in the subsequent paper [6] we will use upper-semicontinuity to compare $\dim T_{\mathbb{Q}}^P$ with $\dim T_{\mathbb{F}_p}^P$.

Recall that if Y_0 is a scheme and $\alpha\colon \mathcal{E}_1 \to \mathcal{E}_2$ is a map of vector bundles on Y_0, the functor $\mathrm{Van.\,Loc.}\,\alpha\colon \mathrm{Sch}^{op} \to \mathrm{Sets}$ given by

$$\mathrm{Van.\,Loc.}\,\alpha(S) = \{t\colon S \to Y_0 \mid t^*\alpha = 0\}$$

is representable, by a closed subscheme of Y_0. If $U = \operatorname{Spec} A$ is an affine open $U \subset Y_0$ on which $\mathcal{E}_1, \mathcal{E}_2$ are trivial, so the map $\alpha\colon A^{r_1} \to A^{r_2}$ on U is given by an $r_2 \times r_1$ matrix (f_{ij}) with entries in A, then $(\mathrm{Van.\,Loc.}\,\alpha) \cap U \hookrightarrow U$ is given by the closed embedding $\operatorname{Spec}(A/(f_{ij})) \hookrightarrow \operatorname{Spec}(A)$. If $F \in \mathbb{Z}[x_0, ..., x_n]_l$ is a homogeneous polynomial of degree l, it gives rise to a map $\beta\colon Ø_{\mathbb{P}^n_{\mathbb{Z}}} \to Ø_{\mathbb{P}^n_{\mathbb{Z}}}(l)$; then the functor $\mathrm{Van.\,Loc.}\,\beta$ is represented by the closed subscheme $V(F) \subset \mathbb{P}^n_{\mathbb{Z}}$.

Let $l \geq 1$ be an integer, and let $V = \mathbb{Z}[x_0, ..., x_n]_l$. For $F \in V$, we can describe the map β above as the composition

$$Ø_{\mathbb{P}^n} \to V \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n} \to Ø_{\mathbb{P}^n}(l),$$

where the first map is given by $F \in V = \Gamma(\mathbb{P}^n_{\mathbb{Z}}, V \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n})$ and the second one is the canonical map.

Let $V' = \mathbb{Z}[x_0, ..., x_n]_{l-1}$. Consider the linear maps $D_i\colon V \to V', F \mapsto \frac{\mathrm{F}}{\mathrm{X}_i}$ for $i = 0, ..., n$, and fix a nonzero polynomial $P \in \mathbb{Q}[z]$. The functor $\mathrm{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(V)\colon \mathrm{Sch}^{op} \to \mathrm{Sets}$ is given as follows: an element of $\mathrm{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(V)(S)$ consists of a closed subscheme $X \hookrightarrow \mathbb{P}^n_S$ such that the composition $X \hookrightarrow \mathbb{P}^n_S \to S$ is flat and each fiber has Hilbert polynomial equal to P, together with a line bundle Ł on S and an injective bundle map $\alpha\colon Ł \hookrightarrow V \otimes_{\mathbb{Z}} Ø_S$.

A map $\alpha\colon Ł \to V \otimes_{\mathbb{Z}} Ø_S$ induces maps $\alpha_i\colon Ł \to V \otimes_{\mathbb{Z}} Ø_S \xrightarrow{D_i \otimes \mathrm{id}} V' \otimes_{\mathbb{Z}} Ø_S$, for $i = 0, ..., n$. Let $\gamma\colon V \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n_{\mathbb{Z}}} \to Ø_{\mathbb{P}^n_{\mathbb{Z}}}(l)$ and $\gamma'\colon V' \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n_{\mathbb{Z}}} \to Ø_{\mathbb{P}^n_{\mathbb{Z}}}(l-1)$ be the canonical maps. Since the pullback to $\mathbb{P}^n_S$ of the target of α coincides with the pullback of the source of γ (similarly for α_i and γ'),

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_S \arrow[r, "r"] \arrow[d, "\pi"] & \mathbb{P}^n_{\mathbb{Z}} \\ & S & \end{tikzcd}$$

we can form the compositions

$$\varepsilon\colon \pi^*Ł \xrightarrow{\pi^*\alpha_0} V \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n_S} \xrightarrow{r^*\gamma} Ø_{\mathbb{P}^n_S}(l)$$
$$\varepsilon_i\colon \pi^*Ł \xrightarrow{\pi^*\alpha_i} V' \otimes_{\mathbb{Z}} Ø_{\mathbb{P}^n_S} \xrightarrow{r^*\gamma'} Ø_{\mathbb{P}^n_S}(l-1),$$

which are maps of line bundles on $\mathbb{P}^n_S$. Thus, for any $(X \hookrightarrow \mathbb{P}^n_S, Ł, \alpha\colon Ł \hookrightarrow V \otimes_{\mathbb{Z}} Ø_S) \in \mathrm{Hilb}^P \times \mathbb{P}(V)(S)$, we have attached maps $\varepsilon, \varepsilon_i, i = 0, ..., n$ of line bundles on $\mathbb{P}^n_S$.

Consider the subfunctor $\mathcal{F}\colon \mathrm{Sch}^{op} \to \mathrm{Sets}$ of the (representabe) functor $\mathrm{Hilb}^P \times \mathbb{P}(V)$, given as follows: $\mathcal{F}(S)$ is the set of all $(X \hookrightarrow \mathbb{P}^n_S, Ł, Ł \hookrightarrow V \otimes_{\mathbb{Z}} Ø_S) \in \mathrm{Hilb}^P \times \mathbb{P}(V)(S)$ such that the pullback of ε and each ε_i (for $i = 0, ..., n$) to X vanishes.

$$\begin{tikzcd} X \arrow[r, hook] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

Proposition 3.1. *The functor $\mathcal{F}$ is representable by a closed subscheme Ω^P of* $\operatorname{Hilb}^P \times \mathbb{P}(V)$.

Proof. Consider the scheme $Y = \operatorname{Hilb}^P \times \mathbb{P}(V)$, and let $(X \hookrightarrow \mathbb{P}^n_Y, \text{Ł}, \alpha\colon \text{Ł} \hookrightarrow V \otimes_k \text{Ø}_Y)$ be the tautological element of $\operatorname{Hilb}^P \times \mathbb{P}(V)(Y)$. This gives rise to maps $\varepsilon, \varepsilon_i$ of line bundles on $\mathbb{P}^n_Y$. Let $\tilde{\varepsilon}, \tilde{\varepsilon}_i$ be the pullbacks of $\varepsilon, \varepsilon_i$ to X.

For a scheme S, $\mathcal{F}(S)$ consists of all maps $S \to Y$ such that the maps of line bundles $\tilde{\varepsilon}, \tilde{\varepsilon}_i$ on X pull back to zero on $X \times_Y S$. Since Y is noetherian and the morphism $X \to Y$ is flat and projective, this functor is representable, by a closed subscheme of Y (see Theorem 5.8 and Remark 5.9 in [4]). ☐

If k is an algebraically closed field and Ω^P_k denotes the basechange $\Omega^P \times \operatorname{Spec} k$, we know the set of closed points of Ω^P_k:

$$\operatorname{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k, \Omega^P_k) = \mathcal{F}(\operatorname{Spec} k).$$

From the definitions, this is just

$$\left\{ (C, [F]) \in \operatorname{Hilb}^P_{\mathbb{P}^n} \times \mathbb{P}(k[x_0, ..., x_n]_l) \mid C \subset V\left(F, \frac{\partial F}{\partial x_i}\right) \right\}$$

(inclusion above denotes scheme-theoretic inclusion).

Corollary 3.2. *Let k be an algebraically closed field, $l \geq 1$ an integer, and $P \in \mathbb{Q}[z]$ a polynomial. The set*

$$\left\{ (C, [F]) \in \operatorname{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(k[x_0, ..., x_n]_l) \mid C \subset V\left(F, \frac{\partial F}{\partial x_i}\right) \right\}$$

is a closed subset of (the set of closed points of) $\operatorname{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(k[x_0, ..., x_n]_l)$.

Let T^P denote the scheme-theoretic image of $\Omega^P \to \mathbb{P}(V)$, so we have a diagram

$$\begin{tikzcd} \Omega^P \arrow[r, hook] \arrow[d] & \operatorname{Hilb}^P \times \mathbb{P}(V) \arrow[d] \\ T^P \arrow[r, hook] & \mathbb{P}(V). \end{tikzcd}$$

Since surjections and closed embeddings are stable under base-change, for any algebraically closed field k, we have a corresponding diagram

$$\begin{tikzcd} \Omega^P_k \arrow[r, hook] \arrow[d] & \operatorname{Hilb}^P_{\mathbb{P}^n_k} \times \mathbb{P}(V_k) \arrow[d] \\ T^P_k \arrow[r, hook] & \mathbb{P}(V_k) \end{tikzcd}$$

(where $V_k = V \otimes_{\mathbb{Z}} k = k[x_0, ..., x_n]_l$) and by looking at closed points, it follows that

$$T^P_k = \{[F] \in \mathbb{P}(V_k) \mid V(F)_{\text{sing}} \text{ contains a subscheme with Hilbert polynomial } P\}.$$

4 Specialization arguments

The main technique that we use in the proof of Theorem 1.1 is a specialization argument, that allows us to bound $\dim\{F \in k[x_0, ..., x_n]_l \mid C \subset V(F)_{\text{sing}}\}$ from above for a fixed C, by degenerating C to a union of linear spaces. In Section 4.1, we prove (for lack of reference) that we can specialize a b-dimensional integral closed subscheme C of $\mathbb{P}^n$ to a union of d b-dimensional linear spaces containing a common $(b-1)$-dimensional linear space. Next, the bound we obtain in Section 4.2 will be the main ingredient for the proof of the main theorem in Section 5.

In this section, k is a fixed algebraically closed field.

4.1 Specialization of a closed subscheme to a union of linear subspaces

The result of this section is known, but we were unable to find a reference, so we include it here.

Let $C \subset \mathbb{P}^n$ be an integral b-dimensional closed subscheme of degree d. Let $P = V(x_0, ..., x_{n-b})$ be the $(b-1)$-dimensional "linear subspace at infinity." Suppose that the linear subspace $H = V(x_{n-b+1}, ..., x_n)$ intersects C in d distinct points Q_i. Let L_i be the unique b-dimensional linear space through P and Q_i. The L_i are distinct because if $L_i = L_j$ for some $i \neq j$, the line through Q_i and Q_j would be contained in H but would have to intersect P; this is impossible, since $P \cap H = \emptyset$. Consider the projective linear transformations

$$A_a = \left(\begin{array}{ccc|ccc} a & & & & & \\ & \ddots & & & & \\ & & a & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

(where the bottom block has size $b \times b$) and let $C_a = A_a C$.

Proposition 4.1. *The underlying topological space of the flat limit $C_0 = \lim_{a \to 0} C_a$ is $\bigcup_{i=1}^d L_i$.*

Proof. Let $C = V(\{G_s\}) \subset \mathbb{P}^n$ (as a scheme), where $G_s \in k[x_0, ..., x_n]$ are homogeneous. Consider the map

$$\sigma \colon \mathbb{P}^n \times (\mathbb{A}^1 - \{0\}) \to \mathbb{P}^n, \quad ([x_0, ..., x_n], a) \mapsto (x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n),$$

and define the closed subscheme $X \subset \mathbb{P}^n \times (A^1 - \{0\})$ as the fiber product

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathbb{P}^n \times (\mathbb{A}^1 - \{0\}) \arrow[d, "\sigma"'] \\ C \arrow[r, hook] & \mathbb{P}^n. \end{tikzcd}$$

In other words,

$$X = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n_{\mathbb{A}^1 - \{0\}},$$

where we regard $G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n) \in k[a, a^{-1}][x_0, ..., x_n]$. This is a flat family $X \to \mathbb{A}^1 - \{0\}$, whose fiber over $a \neq 0$ is C_a (as a subscheme of $\mathbb{P}^n$).

Let $\overline{X}$ be the scheme-theoretic closure of X in $\mathbb{P}^n \times \mathbb{A}^1$. By the proof of Proposition III.9.8 in [5], the flat limit of the family (C_a) is the scheme-theoretic fiber $\overline{X}_0$.

Consider

$$Y = V(G_s(x_0, ..., x_{n-b}, ax_{n-b+1}, ..., ax_n)) \subset \mathbb{P}^n \times \mathbb{A}^1.$$

Then Y is a closed subscheme of $\mathbb{P}^n \times \mathbb{A}^1$ containing X_0 (scheme-theoretically), so Y contains $\overline{X}$. Thus, $\overline{X}_0 \subset Y_0$ is a closed subscheme.

$$\begin{tikzcd}
& & Y \arrow[ddl] \\
X \arrow[r] \arrow[urr] \arrow[d, hook] & \overline{X} \arrow[ur, dotted] \arrow[d, hook] & \\
\mathbb{P}^n \times (\mathbb{A}^1 - \{0\}) \arrow[r, hook] \arrow[d] & \mathbb{P}^n \times \mathbb{A}^1 \arrow[d] & \\
\mathbb{A}^1 - \{0\} \arrow[r, hook] & \mathbb{A}^1 &
\end{tikzcd}$$

We have

$$Y_0 = V(G_s(x_0, ..., x_{n-b}, 0, ..., 0)) \subset \mathbb{P}^n.$$

Thus, as a set, Y_0 is $\bigcup_{i=1}^d L_i$.

We claim that Y_0 is reduced away from P. Equivalently, for $i = 0, .., n-b$, we have to check that $Y_0 \cap D_+(x_i)$ is reduced. To simplify notation, suppose that $i = 0$. Then

$$\begin{aligned}
Y_0 \cap D_+(x_0) &= \operatorname{Spec} \frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_{n-b}, 0, ..., 0))} \\
&= \operatorname{Spec} \left(\frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)} \right) [x'_{n-b+1}, ..., x'_n].
\end{aligned}$$

So we have to show that the 0-dimensional ring

$$\frac{k[x_1, ..., x_n]}{(G_s(1, x_1, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is reduced. We have assumed that C intersects $V(x_{n-b+1}, ..., x_n)$ transversely, so

$$\operatorname{Proj} \frac{k[x_0, ..., x_n]}{(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)}$$

is a reduced 0-dimensional scheme; looking at its intersection with $D_+(x_0)$, we obtain the desired conclusion.

Now that Y_0 is reduced away from a subscheme of smaller dimension, it follows that the Hilbert polynomial of Y_0 has the same degree and leading coefficient (namely, b and $d/b!$, respectively) as the Hilbert polynomial of $(Y_0)_{\text{red}}$. The Hilbert polynomial of the flat limit $\overline{X}_0$ also has degree b and leading coefficient $d/b!$. Moreover, Y_0 is equidimensional, so the inclusion $\overline{X}_0 \hookrightarrow Y_0$ must be a homeomorphism. $\square$

Remark 4.2. The proof above does not imply that Y_0 is reduced everywhere. Let us look at Y_0 in the chart $D_+(x_n)$, so

$$\begin{aligned} Y_0 \cap D_+(x_n) &= \operatorname{Spec} \frac{k[x_0, ..., x_{n-1}]}{(G_s(x_0, ..., x_{n-b}, 0, ..., 0))} \\ &= \operatorname{Spec} \left(\frac{k[x_0, ..., x_n]}{(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)} \right) [x'_{n-b+1}, ..., x'_{n-1}]. \end{aligned}$$

Let $S = k[x_0, ..., x_n]/(G_s(x_0, ..., x_n), x_{n-b+1}, ..., x_n)$. We know that $\operatorname{Proj} S$ is reduced as a scheme by the transversality assumption on $C \cap H$; however, this does not in general imply that S itself is reduced as a ring.

Let $V = k[x_0, ..., x_n]_l$. For each closed subscheme $C \subset \mathbb{P}^n$, define the k-vector space

$$W_C = \{F \in V \mid C \subset V(F)_{\text{sing}}\}.$$

Corollary 4.3. *Let $C \hookrightarrow \mathbb{P}^n$ be an integral closed subscheme of dimension b and degree d. There exist d b-dimensional linear subspaces $L_1, ..., L_d$ of $\mathbb{P}^n$ containing a common $(b-1)$-dimensional linear subspace, such that*

$$\dim W_C \leq \dim W_{\cup L_i},$$

where $\cup L_i$ is given the reduced induced structure.

Proof. Let P be the Hilbert polynomial of C. Recall the incidence correspondence from Corollary 3.2 and apply the upper semicontinuity theorem (see Section 14.3 in [2]) to the map

$$\{(C, [F]) \in \operatorname{Hilb}^P \times \mathbb{P}(V) \mid C \subset V(F)_{\text{sing}}\} \xrightarrow{\pi} \operatorname{Hilb}^P.$$

By Proposition 4.1, $\cup L_i$ (with some scheme structure) is the flat limit C_0 of a family (C_a), with each C_a ($a \neq 0$) being projectively equivalent to $C = C_1$, and hence $\pi^{-1}(C_a) \simeq \pi^{-1}(C)$ for each $a \neq 0$. Therefore,

$$\dim \mathbb{P}(W_C) = \dim \pi^{-1}(C) \leq \dim \pi^{-1}(C_0) = \dim \mathbb{P}(W_{C_0}) \leq \dim \mathbb{P}(W_{\cup L_i}). \qquad \square$$

4.2 An upper bound on the dimension of the space of F such that $C \subset V(F)_{\text{sing}}$, for a fixed C of small degree

Fix a positive integer l. Recall the notation $V = k[x_0, ..., x_n]_l$.

Lemma 4.4. *Let $L \subset \mathbb{P}^n$ be a b-dimensional linear subspace. Then for $F \in V$, we have $L \subset V(F)_{\text{sing}}$ if and only if $F \in I_L^2$. Moreover,*

$$\operatorname{codim}_V \{F \in V \mid L \subset V(F)_{\text{sing}}\} = \binom{l+b}{b} + (n-b)\binom{l-1+b}{b}.$$

Proof. Without loss of generality, $L = V(I)$ with $I = (x_{b+1}, ..., x_n)$. For $F \in V$, we claim that $(F, \frac{\mathrm{F}}{\mathrm{X}_i}) \subset I$ if and only if $F \in I^2$. Suppose that $(F, \frac{\mathrm{F}}{\mathrm{X}_i}) \subset I$. Write $F = F_0 + \sum_{i=b+1}^{n} F_i x_i + T$, where $F_0, F_i \in k[x_0, ..., x_b]$ are homogeneous of degrees $l, l-1$ respectively, and $T \in I_l^2$. Since $\frac{\mathrm{T}}{\mathrm{X}_i} \in I$ for all i, we can assume without loss of generality that $T = 0$. Now, the condition $\frac{\mathrm{F}}{\mathrm{X}_i} \in I$ for $i = b+1, ..., n$ implies $F_i \in I \cap k[x_0, ..., x_b] = 0$, so $F_i = 0$. Then $F = F_0 \in I \cap k[x_0, ..., x_b] = 0$, so $F = 0$ overall, as desired. Clearly, $(S/I^2)_l \simeq k[x_0, ..., x_b]_l \oplus (\bigoplus_{i=b+1}^{n} k[x_0, ..., x_b]_{l-1} x_i)$ has dimension as in the statement. $\square$

Lemma 4.5. *Let $L_1, ..., L_d$ be d b-dimensional linear subspaces of $\mathbb{P}^n$ containing a common $(b-1)$-dimensional linear subspace. Then for $d \le \frac{l+1}{2}$, we have*

$$\operatorname{codim}_V(W_{\cup L_i}) \ge \binom{l+b}{b} + (n-b) \sum_{e=1}^{d} \binom{l-2e+1+b}{b}.$$

Proof. We induct on d. For $d = 1$, we have equality. Assume $2 \le d \le \frac{l+1}{2}$. Assume that the b-dimensional linear subspaces $L_1, ..., L_d$ all contain $P = [0, \underbrace{*, ..., *}_{b}, 0, ..., 0]$ and that none of them is contained in the hyperplane $x_0 = 0$, so the ideal of each of them is of the form $(x_{b+1} - p_{b+1}x_0, ..., x_n - p_n x_0)$ for a uniquely determined tuple $(p_{b+1}, ..., p_n) \in k^{n-b}$. Let

$$I_i = (x_{b+1} - p_{b+1}^{(i)} x_0, x_{b+2} - p_{b+2}^{(i)} x_0, ..., x_n - p_n^{(i)} x_0) \quad \text{for } i = 1, ..., d-1,$$

and without loss of generality

$$I_d = (x_{b+1}, ..., x_n).$$

By Lemma 4.4, $W_{\cup L_i} = (I_1^2 \cap \cdots \cap I_d^2)_l$, so we have to give a lower bound for $\dim(S/I_1^2 \cap \cdots \cap I_d^2)_l$. For $e \in \{d-1, d\}$, let $\mu_e = \dim(S/I_1^2 \cap \cdots \cap I_e^2)_l$. There is a short exact sequence

$$0 \to \left(\frac{I_1^2 \cap \cdots \cap I_{d-1}^2}{I_1^2 \cap \cdots \cap I_d^2}\right)_l \to \left(\frac{S}{I_1^2 \cap \cdots \cap I_d^2}\right)_l \to \left(\frac{S}{I_1^2 \cap \cdots \cap I_{d-1}^2}\right)_l \to 0.$$

So we have to write down enough linearly independent elements in $(I_1^2 \cap \cdots \cap I_{d-1}^2 / I_1^2 \cap \cdots \cap I_d^2)_l$.

For each $i = 1, ..., d-1$, there exists $m_i \in \{b+1, ..., n\}$ such that $p_{m_i}^{(i)} \neq 0$. Let $F = \prod_{i=1}^{d-1} (x_{m_i} - p_{m_i}^{(i)} x_0)^2$. Consider all elements

$$F x_j P(x_0, ..., x_b) \in \left(\frac{I_1^2 \cap \cdots \cap I_{d-1}^2}{I_1^2 \cap \cdots \cap I_d^2}\right)_l,$$

where $j \in \{b+1, ..., n\}$ and $P(x_0, ..., x_b)$ runs through a basis of $k[x_0, ..., x_b]_{l-2d+1}$. Their number is $(n-b)\binom{l-2d+1+b}{b}$ and we claim that they are all linearly independent. Indeed, it suffices to check that their images under the injection $(I_1^2 \cap \cdots \cap I_{d-1}^2 / I_1^2 \cap \cdots \cap I_d^2)_l \hookrightarrow (S/I_d^2)_l$ become linearly independent. This is evident, however, since $(S/I_d^2)_l \simeq k[x_0, ..., x_b]_l \oplus k[x_0, ..., x_b]_{l-1} x_{b+1} \oplus \cdots \oplus k[x_0, ..., x_b]_{l-1} x_n$ as k-vector spaces, and the images of the elements under consideration are

$$(p_{m_1}^{(1)})^2 ... (p_{m_{d-1}}^{(d-1)})^2 x_0^{2(d-1)} x_j P(x_0, ..., x_b).$$

Therefore

$$\mu_d \geq \mu_{d-1} + (n-b)\binom{l-2d+1+b}{b},$$

and the statement follows by induction. $\square$

5 The case of small degree d

With the preparations from the previous section, it is now easy to handle the cases of small degree $2 \leq d \leq \frac{l+1}{2}$ and prove Theorem 1.1. The new ingredient here is a result of Eisenbud and Harris (conjectural for $b \geq 2$), which gives the dimension of the restricted Hilbert scheme. So we can treat the cases of small degree d by applying the theorem on the dimension of fibers to the map $\widetilde{\Omega}^d \to \widetilde{\mathrm{Hilb}}^d$ (Section 5.3). Finally, in Section 6, we perform the analogous calculation for the second largest component of X.

Again, k is a fixed algebraically closed field.

5.1 The component corresponding to $d = 1$

The lemma below is simple, since any two linear b-dimensional subspaces of $\mathbb{P}^n$ are projectively equivalent. Recall the definitions of X^1 and $a_{n,b}(l)$ from the introduction. Let $\mathbb{G}(b,n)$ be the Grassmanian of projective linear b-dimensional subspaces of $\mathbb{P}^n$.

Lemma 5.1. *The set X^1 is an irreducible closed subset of X of dimension equal to $A := \binom{l+n}{n} - a_{n,b}(l)$.*

Proof. Consider

$$\Omega^1 = \{(L,[F]) \in \mathbb{G}(b,n) \times \mathbb{P}(V) \mid L \subset V(F)_{\mathrm{sing}}\} \subset \mathbb{G}(b,n) \times \mathbb{P}(V).$$

By Corollary 3.2, this is a closed subset of the product, since $\Omega^1 = \Omega^P$ with $P(z) = \binom{z+b}{b}$.

Let $\pi\colon \Omega^1 \to \mathbb{G}(b,n)$ and $\rho\colon \Omega^1 \to \mathbb{P}(V)$ denote the two projections. The fiber of π over any linear b-dimensional L is $\mathbb{P}(W_L)$. So Ω^1 is irreducible, and has dimension $\dim \mathbb{P}(W_L) + \dim \mathbb{G}(b,n) = A$ (use Lemma 4.4).

Consider now $\rho\colon \Omega^1 \twoheadrightarrow X^1$. To prove that Ω^1 and X^1 have the same dimension, it suffices to show that some fiber of ρ is 0-dimensional. If we take $L = V(x_0, ..., x_{n-b-1})$, look at $F = \sum_{i=0}^{n-b-2} x_i x_{i+1}$ (in the case $l \geq 3$, which we can tacitly assume). Then L is the only b-dimensional linear subspace contained in $V(F)_{\mathrm{sing}}$. $\square$

5.2 The result of Eisenbud and Harris

We first recall (see [1], p. 3) the following classical result.

Theorem 5.2 (Chow's finiteness theorem)**.** *Fix positive integers n, b, d. There are only finitely many Hilbert polynomials P_α of integral b-dimensional closed subschemes of $\mathbb{P}^n_k$ of degree d. The algebraically closed field k varies as well in this statement.*

Fix k. For an integer $d \geq 1$, let $\mathrm{Hilb}^{b,d}_{\mathbb{P}^n}$ be the disjoint union of the Hilbert schemes $\mathrm{Hilb}^{P_\alpha}_{\mathbb{P}^n}$ for all the finitely many possible Hilbert polynomials P_α of an integral b-dimensional closed subscheme $C \subset \mathbb{P}^n$ of degree d. Define the restricted Hilbert scheme $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ to be the Zariski closure in $\mathrm{Hilb}^{b,d}_{\mathbb{P}^n}$ of the set of integral subschemes, with reduced subscheme structure. Eisenbud and Harris [3] prove the following result for the dimension of $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ in the case $b = 1$.

Theorem 5.3. *Let $b = 1$. For $d \geq 2$, the largest irreducible component of $\widetilde{\mathrm{Hilb}}^{1,d}_{\mathbb{P}^n}$ is the one corresponding to the family of plane curves of degree d; in particular,* $\dim \widetilde{\mathrm{Hilb}}^{1,d}_{\mathbb{P}^n} = 3(n-2) + \frac{d(d+3)}{2}$.

In analogy, for $b \geq 2$, Eisenbud and Harris state the following conjecture:

Conjecture 5.4. For $d \geq 2$, the largest irreducible component of $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ is the one corresponding to the family of degree-d hypersurfaces contained in linear $(b+1)$-dimensional subspaces of $\mathbb{P}^n$; in particular, $\dim \widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n} = (b+2)(n-b-1) - 1 + \binom{d+b+1}{b+1}$.

From now on, we will be assuming that this conjecture holds, so the results we obtain will depend on it, except in the case $b = 1$.

From now on, we fix b and n, and abbreviate $\widetilde{\mathrm{Hilb}}^{b,d}_{\mathbb{P}^n}$ as $\widetilde{\mathrm{Hilb}}^{d}$.

Let Ω^d be the disjoint union of the finitely many Ω^{P_α} (notation as in Proposition 3.1). Also, define T^d as the scheme-theoretic image of $\Omega^d \to \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l)$, so we have a diagram

$$\begin{tikzcd} \Omega^d \arrow[r] \arrow[d] & \mathrm{Hilb}^d \times \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l) \arrow[d] \\ T^d \arrow[r, hook] & \mathbb{P}(\mathbb{Z}[x_0, ..., x_n]_l). \end{tikzcd}$$

For any algebraically closed field k, we have

$$\begin{aligned} T^d_k = \bigcup T^{P_\alpha}_k = \{[F] \in \mathbb{P}(V_k) \mid\ & V(F)_{\mathrm{sing}} \text{ contains} \\ & \text{a subscheme with Hilbert polynomial among } \{P_\alpha\}\}. \end{aligned}$$

Since $X^1 = T^1_k$, we can use X^1 and T^1_k interchangeably.

5.3 The case $d \leq \frac{l+1}{2}$ (small degree)

Fix an integer l as usual, and fix an integer $d > 1$. As usual, let $V = k[x_0, ..., x_n]_l$. Recall that

$$\widetilde{\Omega}^d = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \mid C \subset V(F)_{\mathrm{sing}}\}.$$

Define

$$R^d = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \mid C \text{ is integral, } C \subset V(F)_{\mathrm{sing}}\} \subset \widetilde{\Omega}^d.$$

Let $\overline{R^d}$ be the closure of R^d inside $\widetilde{\Omega}^d$ (or inside $\widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V)$). Let $\pi\colon \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \to \widetilde{\mathrm{Hilb}}^d$ and $\rho\colon \widetilde{\mathrm{Hilb}}^d \times \mathbb{P}(V) \to \mathbb{P}(V)$ denote the first and second projections.

Lemma 5.5. *There exists l_0 (easily computable) such that for all pairs (d,l) with $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$, we have*

$$\dim \overline{R^d} < \dim X^1.$$

It follows that $\dim \rho(\overline{R^d}) < \dim X^1$.

Proof. Let Z be an irreducible component of $\overline{R^d}$. Certainly, $Z \cap R^d \neq \emptyset$, so $\pi(Z)$ contains an integral subscheme $C \subset \mathbb{P}^n$. Degenerate C to a union $\bigcup_{i=1}^d L_i$ of d b-dimensional linear spaces, as in Section 4.1. Let L_0 be any linear b-dimensional subspace of $\mathbb{P}^n$. By abuse of notation, let $\pi \colon Z \twoheadrightarrow \pi(Z) \subset \widetilde{\mathrm{Hilb}}^d$. By the theorem on the dimension of fibers, we have

$$\begin{aligned}
\dim Z &\leq \dim \pi^{-1}(C) + \dim \pi(Z) \\
&\leq \dim \mathbb{P}(W_C) + \dim \pi(Z) \\
&\leq \dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d. \qquad (1)
\end{aligned}$$

Thus, it suffices to check that

$$\dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d < \dim \mathbb{P}(W_{L_0}) + (b+1)(n-b)$$

(recall Lemma 5.1), or, equivalently, that

$$\mathrm{codim}_V\, W_{L_0} + \dim \widetilde{\mathrm{Hilb}}^d < \mathrm{codim}_V\, W_{\cup L_i} + (b+1)(n-b).$$

By Lemmas 4.4 and 4.5, it suffices to prove the inequality

$$\begin{aligned}
&\binom{l+b}{b} + (n-b)\binom{l-1+b}{b} + \dim \widetilde{\mathrm{Hilb}}^d \\
&< \binom{l+b}{b} + (n-b)\sum_{e=1}^{d}\binom{l-2e+1+b}{b} + (b+1)(n-b),
\end{aligned}$$

or, equivalently,

$$\dim \widetilde{\mathrm{Hilb}}^d - (b+1)(n-b) < (n-b)\sum_{e=2}^{d}\binom{l-2e+1+b}{b}, \qquad (2)$$

for all $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$. Let $c = (b+2)(n-b-1) - 1 - (b+1)(n-b)$. Assume Conjecture 5.4; then (2) is equivalent to

$$c + \binom{d+b+1}{b+1} < (n-b)\sum_{e=2}^{d}\binom{l-2e+1+b}{b} \qquad (3)$$

for all $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$.

For $l \geq 2d-1$, the right hand side of (3) is at least

$$\begin{aligned}
(n-b)\sum_{e=2}^{d}\binom{2d-2e+b}{b} &= (n-b)\sum_{k=0}^{d-2}\binom{2k+b}{b} \qquad \text{(where } k = d-e\text{)} \\
&= (n-b)\sum_{k=0}^{d-2}\frac{(2k+b)(2k+b-1)...(2k+1)}{b!} \\
&= (n-b)\sum_{k=0}^{d-2}\left(\frac{2^b k^b}{b!} + \dots\right).
\end{aligned}$$

Recall that $\sum_{k=0}^{d} k^b$ is a polynomial in d of degree $b+1$ and leading coefficient $\frac{1}{b+1}$; so the right hand side of (3) dominates a polynomial in d of degree $b+1$ and leading coefficient $(n-b)\frac{2^b}{b!}\frac{1}{b+1} = \frac{(n-b)2^b}{(b+1)!}$. Since $\binom{d+b+1}{b+1}$ is a polynomial in d of the same degree $b+1$, but smaller leading coefficient $\frac{1}{(b+1)!}$, the inequality (3) holds for all $l \geq 2d-1$ and all $d > d_0$ for some d_0 (which is easy to calculate algorithmically, for fixed n, b).

On the other hand, for each fixed value $d = 2, ..., d_0$, the right hand side of (3) is a polynomial in l of degree b and positive leading coefficient $\frac{(n-b)(d-1)}{b!}$, while the left hand side is a constant. So there is l_0 (easily computable for given b, n, d_0) such that for all $d = 2, ..., d_0$ and $l \geq l_0$, the inequality (3) holds true. Therefore, for all $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$, the inequality from the statement of the lemma holds, as well. $\square$

Let l_0 be as in Lemma 5.5.

Corollary 5.6. *Let $2 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$. If $Z \subset T_k^d$ is an irreducible component, then either $Z = X^1$, or $\dim Z < \dim X^1$.*

Proof. We claim that if $[F] \in T_k^d - \left(T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'})\right)$, then $V(F)_{\text{sing}}$ contains an integral b-dimensional subscheme of degree d. Indeed, $V(F)_{\text{sing}}$ contains some integral b-dimensional closed subscheme of degree $\tilde{d} \in \{1, ..., d\}$; if $[F] \notin \cup_{d'=1}^{d-1} T_k^{d'}$, then necessarily $\tilde{d} = d$.

Now, we can induct on d, so assume that $Z \nsubseteq \cup_{d'=1}^{d-1} T_k^{d'}$. Note that $Z - \left(Z \cap (\cup_{d'=1}^{d-1} T^{d'})\right) \subset Z$ is a dense open subset of Z, which therefore has the same dimension as Z, but is contained in $T_k^d - \left(T_k^d \cap (\cup_{d'=1}^{d-1} T_k^{d'})\right) \subset \rho(R^d) \subset \rho(\overline{R^d})$. Thus $\dim Z \leq \dim \rho(\overline{R^d}) < \dim X^1$, by Lemma 5.5 $\square$

This completes the proof of Theorem 1.1.

We can obtain a weaker version that does not rely on the conjecture of Eisenbud and Harris:

Lemma 5.7. *Fix an integer B. There exists l_0 such that for all $2 \leq d \leq B$ and $l \geq l_0$, for any irreducible component Z of T_k^d, either $Z = X^1$, or $\dim Z < \dim X^1$.*

Proof. Just note that inequality (2) in the proof of the previous lemma is satisfied when $d \in \{2, ..., B\}$ is fixed and $l \gg 0$. $\square$

6 On the second largest component of X

6.1 The existence of a component of X of the expected second-largest dimension

In contrast to the treatment of the largest component of X, the existence of a component of the expected second-largest dimension is a little more subtle, so there will be an extra twist in the argument.

Again, k is any algebraically closed field.

We begin with the following preparation. Consider a b-dimensional closed subscheme $C = V(f, x_{b+2}, ..., x_n)$ of $\mathbb{P}^n$, where $f \in k[x_0, ..., x_{b+1}]_d - \{0\}$, and set $W = (f, x_{b+2}, ..., x_n)_l^2$.

Lemma 6.1. *Assume $l \geq 2d+1$. There is a dense open subset $U_1 \subset \mathbb{P}(W)$ such that for all $[F] \in U_1$, $V(F)_{\text{sing}} = C$ (set-theoretically).*

Proof. Consider the incidence correspondence

$$Y_1 = \{([F], P) \in \mathbb{P}(W) \times (\mathbb{P}^n - C) \mid P \in V(F)_{\text{sing}}\} \subset \mathbb{P}(W) \times (\mathbb{P}^n - C)$$

(it is a closed subset of this product, and hence a quasiprojective variety). We are going to show that $\dim Y_1 < \dim \mathbb{P}(W)$; this will imply that the closure $\overline{Y_1}$ of Y_1 in $\mathbb{P}(W) \times \mathbb{P}^n$ also has dimension smaller than that of $\mathbb{P}(W)$, and thus the image of this closure under the projection to $\mathbb{P}(W)$ will be a proper closed subset of $\mathbb{P}(W)$. Its complement U_1 will satisfy the condition of the lemma.

Consider the second projection $\tau\colon Y_1 \to \mathbb{P}^n - C$, and let $P \in \mathbb{P}^n - C$. We claim the fiber $\tau^{-1}(P)$ is a projective linear subspace of $\mathbb{P}(W)$ of codimension $n+1$. This will imply that Y_1 is irreducible, of dimension $\dim Y_1 = \dim \mathbb{P}(W) - 1$.

Suppose first that $P \in \cup_{i=b+2}^n D_+(x_i)$. Without loss of generality, assume that $P = [a_0, ..., a_{n-1}, 1]$. Notice that $\tau^{-1}(P)$ is just

$$\mathbb{P}\left(((x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2)_l\right) \subset \mathbb{P}(W),$$

so it remains to show that

$$\dim\left(\frac{W}{(x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2}\right)_l = n+1,$$

i.e., that the map

$$\left(\frac{(f, x_{b+2}, ..., x_n)^2}{(x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2}\right)_l \hookrightarrow \left(\frac{S}{(x_0 - a_0x_n, ..., x_{n-1} - a_{n-1}x_n)^2}\right)_l \simeq k[x_n]_l \oplus \left(\bigoplus_{i=0}^{n-1} k[x_n]_{l-1}(x_i - a_ix_n)\right)$$

is an isomorphism. The images of x_n^l and $x_n^{l-1}(x_i - a_ix_n)$ for $i = 0, ..., n-1$ give a basis of the target.

Suppose now that $P \in V(x_{b+2}, ..., x_n)$, without loss of generality $P = [1, a_1, ..., a_{b+1}, 0, ..., 0]$. As above, we have to prove that the following map is an isomorphism:

$$\left(\frac{(f, x_{b+2}, ..., x_n)^2}{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0, x_{b+2}, ..., x_n)^2 \cap (f, x_{b+2}, ..., x_n)^2}\right)_l \hookrightarrow \left(\frac{S}{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0, x_{b+2}, ..., x_n)^2}\right)_l \simeq k[x_0]_l \oplus \left(\bigoplus_{i=1}^{b+1} k[x_0]_{l-1}(x_i - a_ix_0)\right) \oplus \left(\bigoplus_{i=b+2}^{n} k[x_0]_{l-1}x_i\right).$$

Now, dehomogenize f with respect to x_0, consider a Taylor expansion at $(a_1, ..., a_{b+1})$, and homogenize to degree l again, so $f \equiv ax_0^d \pmod{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0)}$ with $a \neq 0$. So $f^2 \equiv a^2x_0^{2d} \pmod{(x_1 - a_1x_0, ..., x_{b+1} - a_{b+1}x_0)}$. Now, the elements $f^2x_0^{l-2d-1}(x_i - a_ix_0)$ (for $i = 1, ..., b+1$), $f^2x_0^{l-2d-1}x_i$ (for $i = b+2, ..., n$), and $f^2x_0^{l-2d}$ map to a basis of the target. $\square$

Now, fix n, b as usual, and let $d \geq 1$. Define

$$\begin{aligned}\beta_d(l) &= \binom{l+b+1}{b+1} - \binom{l-2d+b+1}{b+1} + (n-b-1)\left(\binom{l+b}{b+1} - \binom{l-d+b}{b+1}\right)\\ &= \frac{(n-b+1)d}{b!} l^b + \dots\end{aligned}$$

Let $I = (f, x_{b+2}, ..., x_n) \subset S = k[x_0, ..., x_n]$, where $f \in k[x_0, ..., x_{b+1}]_d - \{0\}$. Consider the composition

$$\Phi \colon k[x_0, ..., x_{b+1}]_l \oplus \left(\bigoplus_{i=b+2}^{n} k[x_0, ..., x_{b+1}]_{l-1} x_i \right) \hookrightarrow S_l \twoheadrightarrow S_l/(I^2 \cap S_l).$$

Note that Φ is surjective.

Lemma 6.2. *We have that*

$$\ker(\Phi) = \{P + \sum_{i=b+2}^{n} P_i x_i \ : \ f^2 | P, f | P_i \text{ for } i = b+2, ..., n\}.$$

For $l \geq 2d$, the codimension of I_l^2 in S_l equals $\beta_d(l)$.

Proof. If $P + \sum P_i x_i \in \ker(\Phi)$, then we can write $P + \sum P_i x_i = T \in I^2$. Expand both sides as polynomials in $x_{b+2}, ..., x_n$ and just compare the two expressions. The second part is an immediate consequence. □

Lemma 6.3. *Let $C \hookrightarrow \mathbb{P}^n$ be any integral b-dimensional closed subscheme of degree 2, with (saturated) ideal I. If $F \in k[x_0, ..., x_n]_l$ satisfies $C \subset V(F)_{\text{sing}}$, then $F \in I_l^2$.*

Proof. Projection from a point on C shows that C is contained in a linear $(b+1)$-dimensional subspace of $\mathbb{P}^n$. So we can assume that $C = V(I)$, with $I = (f, x_{b+2}, ..., x_n)$, where $f \in k[x_0, ..., x_{b+1}]_2 - \{0\}$ is irreducible. We claim that the ideal I^2 is saturted. Indeed, let $F \in S$ be homogeneous, and suppose that $x_j^M F \in I^2$ for all $j = 0, ..., n$ (and for some M). Write $F = P + \sum_{i=b+2}^{n} P_i x_i + T$, where $P, P_i \in k[x_0, ..., x_{b+1}]$ are homogeneous of the appropriate degrees, and $T \in (x_{b+2}, ..., x_n)^2$. Since $x_0^M F \in I^2$, Lemma 6.2 implies that $f^2 | x_0^M P$ and $f | x_0^M P_i$ for each $i = b+2, ..., n$. Since f and x_0 are relatively prime, it follows that $f^2 | P$ and $f | P_i$ for each i, and hence $F \in I^2$.

Since C is a local complete intersection and the ideal I^2 is saturated, the conclusion now follows from Corollary 2.3 in [7]. □

Let $P = \binom{z+b+1}{b+1} - \binom{z-1+b}{b+1}$ (this is the Hilbert polynomial of a degree-2 hypersurface in $\mathbb{P}^{b+1}$). Recall that $\widetilde{\text{Hilb}}^P$ denotes the closure in Hilb^P of the set of integral b-dimensional closed subschemes of degree 2; in this case, a point in $\widetilde{\text{Hilb}}^P$ is, up to a change of coordinates, a closed subscheme of the form $V(f, x_{b+2}, ..., x_n) \subset \mathbb{P}^n$, where $f \in k[x_0, .., x_{b+1}]_2 - \{0\}$ (not necessarily irreducible of course). Note that

$$\begin{aligned}\dim \widetilde{\text{Hilb}}^P &= \dim \mathbb{G}(b+1, n) + \dim \mathbb{P}(k[x_0, ..., x_{b+1}]_2)\\ &= (b+2)n - \frac{b(b+1)}{2}. \end{aligned} \tag{4}$$

By Lemma 6.2, if $f \in k[x_0, ..., x_{b+1}]_2 - \{0\}$, then

$$\dim \mathbb{P}\left((f, x_{b+2}, ..., x_n)_l^2\right) = \binom{l+n}{n} - \beta_2(l) - 1. \tag{5}$$

Recall the usual incidence correspondence (where inclusion is scheme-theoretic)

$$\widetilde{\Omega}^P = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^P \times \mathbb{P}(V) \mid C \subset V(F)_{\mathrm{sing}}\} \subset \widetilde{\mathrm{Hilb}}^P \times \mathbb{P}(V).$$

Recall that π and ρ denote the projections to $\widetilde{\mathrm{Hilb}}^P$ and $\mathbb{P}(V)$, respectively. For $C \subset \mathbb{P}^n$ a closed subscheme, let I_C denote its (saturated) ideal. Consider the subset

$$Z' = \{(C, [F]) \in \widetilde{\mathrm{Hilb}}^P \times \mathbb{P}(V) \mid F \in I_C^2\} \subset \widetilde{\Omega}^P.$$

Lemma 6.4. *The subset Z' of $\widetilde{\Omega}^P$ is irreducible.*

Proof. By Lemma 6.2, for a fixed $f \in k[x_0, ..., x_{b+1}]_2 - \{0\}$ and given $F = F_0 + \sum_{i=b+2}^n F_i x_i + T \in k[x_0, ..., x_n]_l$, where $F_0 \in k[x_0, ..., x_{b+1}]_l$, $F_i \in k[x_0, ..., x_{b+1}]_{l-1}$, and $T \in (x_{b+2}, ..., x_n)_l^2$, we have that $F \in (f, x_{b+2}, ..., x_n)_l^2$ if and only if $f^2 | F_0$ and $f | F_i$ for each $i = b+2, ..., n$.

Let $V' = k[x_0, ..., x_{b+1}]_{l-4} \oplus \left(\bigoplus_{i=b+2}^n k[x_0, ..., x_{b+1}]_{l-3}\right) \oplus (x_{b+2}, ..., x_n)_l^2$. Denote by $\mathbb{A}(k[x_0, ..., x_{b+1}]_2)$ the affine space parametrizing points in $k[x_0, ..., x_{b+1}]_2$. Consider the composition

$$\begin{array}{c} \mathrm{Aut}(\mathbb{P}^n) \times (\mathbb{A}(k[x_0, ..., x_{b+1}]_2) - \{0\}) \times \mathbb{P}(V') \\ \big\downarrow \\ \mathrm{Aut}(\mathbb{P}^n) \times \mathbb{P}(k[x_0, ..., x_{b+1}]_2) \times \mathbb{P}(V) \\ \big\downarrow \\ \widetilde{\mathrm{Hilb}}^P \times \mathbb{P}(V) \end{array}$$

where the first map is given by

$$(\sigma, f, [Q, R_{b+2}, ..., R_n, T]) \longmapsto (\sigma, [f], [f^2 Q + \sum_{i=b+2}^n f R_i x_i + T])$$

and the second map is given by

$$(\sigma, [f], [F]) \longmapsto (V(f^\sigma, x_{b+2}^\sigma, ..., x_n^\sigma), [F]^\sigma).$$

By construction, Z' is precisely the image of the composition, hence is irreducible. $\square$

Remark 6.5. It is not true that the fibers of $\widetilde{\Omega}^P \xrightarrow{\pi} \widetilde{\mathrm{Hilb}}^P$ are all of the same dimension. For example, let $b = 1, n = 3$, and look at $C = V(x_2^2, x_3) \in \widetilde{\mathrm{Hilb}}^P$. Let $F = x_2^3 x_0^{l-3}$. Then $(C, [F]) \in \pi^{-1}(C)$, but $F \notin (x_2^2, x_3)^2$. This is why we have to study the auxiliary Z'.

Let Z be the closure of Z' in $\widetilde{\Omega}^P$.

Lemma 6.6. *We have that*

$$\dim Z = \binom{l+n}{n} - \beta_2(l) - 1 + (b+2)n - \frac{b(b+1)}{2}.$$

Proof. First, $\pi(Z') = \widetilde{\mathrm{Hilb}}^P$, since given any $C \in \widetilde{\mathrm{Hilb}}^P$, the ideal I_C^2 contains forms of degree 4 already, so we can certainly find $F \in (I_C^2)_l$. Thus, $\pi\colon Z \twoheadrightarrow \widetilde{\mathrm{Hilb}}^P$ is onto. A generic $C \in \widetilde{\mathrm{Hilb}}^P$ is an integral b-dimensional closed subscheme of degree 2; for such a C, by Lemma 6.3, we know $Z'_C = \widetilde{\Omega}_C^P$ and hence also $Z_C = Z'_C$. This allows us to compute $\dim Z_C = \dim Z'_C = \binom{l+n}{n} - \beta_2(l) - 1$. This computes $\dim Z = \dim \widetilde{\mathrm{Hilb}}^P + \dim Z_C$ and gives the desired result, by virtue of (4) and (5). $\square$

Lemma 6.7. *$X^2 := \rho(Z)$ is an irreducible closed subset of X of dimension $\binom{l+n}{n} - \beta_2(l) - 1 + (b+2)n - \frac{b(b+1)}{2}$. If $[F] \in X$ contains an integral closed subscheme of dimension b and degree 2 in its singular locus, then $[F] \in X^2$.*

Proof. It is clear that $\rho(Z)$ is an irreducible closed subset of X, since Z is irreducible and closed in $\widetilde{\Omega}^P$. Choose any integral b-dimensional C of degree 2. Apply Lemma 6.1 to C to find $[F] \in \mathbb{P}(V)$ such that we have a homeomorphism $C \hookrightarrow V(F)_{\mathrm{sing}}$. If $\hat{C} \in \widetilde{\mathrm{Hilb}}^P$ is another closed subscheme contained in $V(F)_{\mathrm{sing}}$, then necessarily we have $C \hookrightarrow \hat{C}$, since C is reduced. Hence $C = \hat{C}$, since C and $\hat{C}$ have the same Hilbert polynomial. Therefore, the map $Z \to \rho(Z)$ has a 0-dimensional fiber, so $\dim \rho(Z) = \dim Z$.

Let $[F] \in X$ be such that $V(F)_{\mathrm{sing}}$ contains an integral b-dimensional closed subscheme C of $\mathbb{P}^n$ of degree 2. Then we know that $F \in I_C^2$ by Lemma 6.3, so $(C, [F]) \in Z'$, and hence in fact $[F] \in \rho(Z') \subset \rho(Z) = X^2$. $\square$

6.2 The analogue of Theorem 1.1 regarding the second-largest component

Here we discuss a calculation similar to the one in section 5.3, which addresses the question of the second largest component of X.

Note that

$$\beta_2(l) = \binom{l+b+1}{b+1} - \binom{l+b-3}{b+1} + (n-b-1)\left(\binom{l+b}{b+1} - \binom{l+b-2}{b+1}\right)$$

and set $\gamma_2(l) = \beta_2(l) + 1 - (b+2)n + \frac{b(b+1)}{2}$. We know that $\binom{l+n}{n} - \gamma_2(l)$ is the dimension of X^2. We are still assuming Conjecture 5.4.

Lemma 6.8. *There exists l_0 (easily computable) such that for all pairs (d, l) with $3 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$ (if $b = n-1$, assume $d \geq 4$), and any irreducible component Z of T_k^d, either $Z \subset T_k^1 \cup T_k^2$, or*

$$\dim Z < \binom{l+n}{n} - \gamma_2(l).$$

(In the case $b = n-1$, we can describe X explicitly, so this case is not of interest to us.)

Proof. Precisely as in Lemma 5.5, because of inequality (1), it suffices to establish the inequality

$$\dim \mathbb{P}(W_{\cup L_i}) + \dim \widetilde{\mathrm{Hilb}}^d < \binom{l+n}{n} - \gamma_2(l), \qquad \text{i.e.,}$$

$$\gamma_2(l) - 1 + \dim \widetilde{\mathrm{Hilb}}^d < \mathrm{codim}_V(W_{\cup L_i}).$$

Set $c = -\frac{(b+1)(b+4)}{2} - 1$. By Lemma 4.5 and Conjecture 5.4, we are reduced to proving that

$$c + \beta_2(l) + \binom{d+b+1}{b+1} < \binom{l+b}{b} + (n-b)\sum_{e=1}^{d}\binom{l-2e+1+b}{b},$$

or, equivalently, that

$$c + (n-b)\binom{l+b-3}{b-1} + \binom{l+b-3}{b} + \binom{d+b+1}{b+1} < (n-b)\sum_{e=3}^{d}\binom{l-2e+1+b}{b}. \quad (6)$$

Suppose first that $n-b>1$. If $d=3$, this inequality is certainly satisfied for $l \gg 0$ (look at the leading terms of both sides). Consider now $d \geq 4$. Since $n-b>1$, we can find l' such that for all $l \geq l'$,

$$c + (n-b)\binom{l+b-3}{b-1} + \binom{l+b-3}{b} < (n-b)\binom{l-5+b}{b}.$$

What is left now is to prove that there exists l'' such that for $l \geq l''$ and $4 \leq d \leq \frac{l+1}{2}$, we have

$$\binom{d+b+1}{b+1} < (n-b)\sum_{e=4}^{d}\binom{l-2e+1+b}{b}.$$

This is analogous to (3) and follows exactly as in the proof of Lemma 5.5. Now we just take $l_0 = \max(l', l'')$.

Suppose now that $n-b=1$ and $d \geq 4$. If $d=4$, inequality (6) certainly holds for large l (the leading term of the right hand side is $\frac{2l^b}{b!}$). Consider $d \geq 5$. We can find l' such that for all $l \geq l'$,

$$c + \binom{l+b-3}{b-1} + \binom{l+b-3}{b} < \binom{l-5+b}{b} + \binom{l-7+b}{b}.$$

Finally, we have to show that that there exists l'' such that for $5 \leq d \leq \frac{l+1}{2}$ and $l \geq l''$, we have

$$\binom{d+b+1}{b+1} < \sum_{e=5}^{d}\binom{l-2e+1+b}{b}.$$

Again, this is analogous to inequality (3). □

In [6], we will use the above result to show that when $p = \mathrm{char} k > 0$, there exists (again, effectively computable) $l_0 = l_0(n, b, p)$, such that for $l \geq l_0$, X^2 is the unique irreducible component of X of second largest dimension.

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